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Searching for acyclic orientations of graphs

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Abstract

We want to find an unknown acyclic orientation $\mathcal{O}^*$ of an (undirected) graph G by testing for certain edges how they are oriented according to $\mathcal{O}^*$. How many tests do we need in the worst case? We give upper and lower bounds for this number $c(G)$ in terms of the independence number of G and study the class of exhaustive graphs, i.e. graphs satisfying $c(G) = |E(G)|$. It is shown that there exist nonexhaustive graphs with arbitrarily large girth. The extremal exhaustive graphs are determined for $n \geqslant 7$.

1. Introduction

Suppose $G = (V, E)$ is a simple graph and imagine two players, A ('Algy') and S ('Strategist') playing the following game on G: A asks S questions about some hypothetical acyclic orientation $\mathcal{O}^*$ of G which he wants to identify. For each edge $xy \in E(G)$, A is allowed to ask whether xy is directed from x to y or vice versa. While A wants to find $\mathcal{O}^*$ by asking as few questions as possible, S provides answers to the questions in order to maximize the number of probed edges. Player S is not supposed to fix an orientation in advance, but his answers must be consistent. The game is over, of course, as soon as there is only one acyclic orientation $\mathcal{O}^*$ which is compatible with all the answers. Denote by $c(G)$ the number of questions asked in the game if both players play optimally from their point of view. Then $c(G)$ is the worst case complexity for the search problem with search domain $A(G) := \{\mathcal{O}: \mathcal{O}$ is an acyclic orientation of $G\}$ and admissible tests 'Is $xy \in E(G)$ directed from x to y or from y to x?' (See [7]). As an example, consider the complete graph K_n. There are exactly $n!$ acyclic orientations of K_n which canonically induce the $n!$ different total orderings of the vertices. Testing

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an edge means to determine which endpoint is larger in the corresponding total order. Hence, for complete graphs, we recover the problem of *minimum comparison sorting* and conclude that $c(K_n) \sim n \ln n$ $(n \to \infty)$. As was noted in [1, p. 323], $c(G)$ is closely related to a problem arising from graph coloring which was introduced by Manber and Tompa [10]: Given some function f from the vertices of G to the real numbers, how many tests of type 'Is $f(u) = f(v)$?' $(uv \in E(G))$ are needed to find out whether f is a proper coloring of G? It is easy to see that this number equals $c(G)$ (see [1, Exercise 6.3.7]), but we are not going to use this second definition. In what follows, we derive upper and lower bounds for $c(G)$ and study some properties of *exhaustive* graphs, i.e. graphs satisfying $c(G) = |E(G)|$. Throughout this paper, log and ln denote logarithms with base 2 and e, respectively. For undefined graph theoretical notions, we refer the reader to [6].

2. Upper and lower bounds

Denote by $a(G)$ the number of acyclic orientations of G. Then the usual information theoretic lower bound (see [1, p. 24]) reads as follows.

Proposition 1. *For all graphs G, $c(G) \geqslant \lceil \log a(G) \rceil$.*

Remarks. As noted above, $c(K_n) \sim n \log n \sim \log a(K_n)$, but in general the quotient $c(G)/\log a(G)$ may become arbitrarily large. To see this, let G be the complete, bipartite graph with color classes U and W, $|U| = |W| = n$. Imagine player S to answer according to the following strategy: *Direct every edge from U to W.* It is easy to see that in this case player A has to probe all the n^2 edges of G, hence $c(G) = n^2$. On the other hand, $a(G) \leqslant (2n)!$, since each acyclic orientation of G induces a partial order on the vertices which can be embedded into a linear order, hence $\log a(G) \leqslant (2n)\log(2n)$ from which our claim follows.

(ii) An exact formula for $a(G)$ was given by Stanley [13]: He showed that $a(G) = |p_G(-1)|$, where $p_G(x)$ denotes the chromatic polynomial of G. However, the computation of $p_G(x)$ is an NP-hard problem. Moreover, Linial [9] showed that it is $\# P$-hard to calculate $a(G)$.

(iii) Linial also gave the following lower bound on $a(G)$: If $|E(G)| = \binom{k}{2} + l$, $0 \leqslant l < k$, then $a(G) \geqslant k!(l+1)$ [8].

The following result gives bounds for $c(G)$ in terms of the independence number $\alpha = \alpha(G)$ of G.

Theorem 2. *For all graphs G on n vertices and with independence number α, the* following inequalities hold:

$$n \log \frac{n}{\alpha} - n \log e + \frac{1}{2}(2 + \log n) < c(G) < \alpha n \left(\log \frac{n}{\alpha} + 1\right).$$

Proof. *Lower bound*: We estimate $a(G)$ from below and apply Proposition 1: Each of the $n!$ linear orderings of $V(G)$ defines precisely one acyclic orientation of G (by 'restriction'). We show that each acyclic orientation $\mathcal{O}$ of G can have at most α^n linear extensions. In fact, the smallest vertex in such a linear order has to be a source (vertex of in-degree 0) in the directed graph $\vec{G}$ which arises from G by orienting the edges according to $\mathcal{O}$. Since the sources form an independent set, there are at most α choices for the first vertex. Deleting it from G, the independence number cannot increase and our claim follows by induction. Consequently, by Stirling's formula,

$$c(G) \geqslant \log a(G) \geqslant \log \frac{n!}{\alpha^n} > n \log \frac{n}{e\alpha} + \frac{1}{2}\log(2\pi n)$$

$$\geqslant n \log \frac{n}{\alpha} - n \log e + \frac{1}{2}(2 + \log n).$$

Upper bound: We apply induction on n. For small n the assertion is trivial. Now choose $v \in V(G)$ arbitrarily and suppose that player A first finds the acyclic orientation of $G - v$ by asking $c(G - v)$ questions. Let θ denote the minimum number of directed paths whose vertex-disjoint union is $V(G)\setminus\{v\}$. By the theorem of Gallai and Milgram [5], $\theta \leqslant \alpha$. Decompose $G - v$ into vertex-disjoint paths $P_1, \ldots, P_\theta$. Denoting by m_i the number of edges joining v to P_i, we can find the orientation of those edges by $\lceil \log m_i \rceil < \log m_i + 1$ questions by binary search. (Recall that P_i is a linear order.) Thus, by Jensen's inequality:

$$c(G) < c(G - v) + \sum_{i=1}^{\theta} (\log m_i + 1) \leqslant c(G - v) + \theta\left(1 + \log \frac{\sum_{i=1}^{\theta} m_i}{\theta}\right)$$

$$< c(G - v) + \theta\left(1 + \log \frac{n}{\theta}\right) \leqslant c(G - v) + \alpha + \alpha \log \frac{n}{\alpha}.$$

Since $c(G - v) < \alpha(n - 1)(\log(n/\alpha) + 1)$ by the induction hypothesis, the result follows. $\square$

As a corollary, we obtain the answer to the following question of Erdős [4]: How small should the number $\binom{n}{2} - |E(G)|$ be in order to ensure $c(G) = o(n^2)$?

Corollary 3. *If* $\binom{n}{2} - |E(G)| = o(n^2)$, *then* $c(G) = o(n^2)$ *as* $n \to \infty$. *On the other hand, for every* $0 < \varepsilon < \frac{1}{2}$ *there is some* $\delta > 0$ *such that for large n there exists a graph G with at least* $(\frac{1}{2} - \varepsilon)n^2$ *edges satisfying* $c(G) \geqslant \delta n^2$.

Proof. (i) The assumption $\binom{n}{2} - |E(G)| = o(n^2)$ implies $\alpha(G) = o(n)$, and therefore $\alpha \log(n/\alpha) = o(n)$ also holds. Hence $c(G) = o(n^2)$ by the upper bound in Theorem 2.

(ii) For the converse, first note the following very general observation: If H is an induced subgraph of G, then $c(G) \geqslant c(H)$. Indeed, player S can choose an acyclic orientation of $G - E(H)$ where all edges between $V(H)$ and its complement are directed from $V(H)$ to $V(G)\setminus V(H)$ and tell player A from the beginning how the edges of $E(G)\setminus E(H)$ are oriented. Player A will need $c(H)$ further questions if S uses an optimal strategy on H. Now delete the edges of two disjoint copies of $K_{\lceil \varepsilon' n \rceil}$ from K_n, where $0 < \varepsilon'^2 < \varepsilon$. The resulting graph has at least

$$\binom{n}{2} - \frac{1}{2}(\varepsilon' n + 1)\varepsilon' n = n^2\left(\frac{1}{2} - \varepsilon'^2\right) - \frac{n}{2} + \varepsilon' n$$

edges and contains $K_{\lceil \varepsilon' n \rceil, \lceil \varepsilon' n \rceil}$ as an induced subgraph, hence its complexity is at least $(\varepsilon' n)^2$. □

As another consequence of Theorem 2 we can determine the number $c(G)$ for a random graph with fixed edge probability within a factor of order $\log n$. (For the basic facts about random graphs, see [2].)

Corollary 4. *Let p be a fixed real, $0 < p < 1$. Suppose that $G = G(n, p)$ is a random graph on n vertices whose edges are drawn independently with probability p and denote by Pr the probability measure in this random graph model. Then there is some constant $\gamma = \gamma(p)$ such that (for $n \to \infty$)*

$$Pr(n \log n - O(n \log\log n) < c(G) < \gamma n \log^2 n + O(n \log n \log\log n)) \to 1.$$

Proof. This follows from Theorem 2 by using the well-known fact that

$$\alpha(G) < \left(\frac{2}{\log 1/(1-p)} + \varepsilon\right)\log n$$

with probability tending to 1 for all fixed $\varepsilon > 0$ (see [2, Ch. XI]). □

A closer examination of the proof of the upper bound in Theorem 2 shows:

Corollary 5. *There is a polynomial algorithm by which Algy can design a sequence of at most $\alpha n(\log(n/\alpha) + 1)$ questions which determine the orientation of G.*

Proof. It suffices to show that a decomposition of $G - v$ into disjoint paths à la Gallai–Milgram can be found in polynomial time. But this is clear since it is well-known that such a decomposition can be found by network flow techniques (see [3, p. 142, Problem 6.1]). □

3. Exhaustive graphs

Recall that a graph is called *exhaustive* if and only if $c(G) = |E|$, i.e., if player A can be forced to probe all the edges. Consider the following strategy for player S: Choose some fixed acyclic orientation $\mathcal{O}$ of G and answer the questions of A according to that orientation. We denote by $c(G; \mathcal{O})$ the minimum number of questions which A needs to determine the orientation (unknown for him).

Proposition 6. *Suppose G is a graph and $\mathcal{O}$ an acyclic orientation of G. Denote by $G_{\mathcal{O}}$ the oriented graph arising from G if all the edges are oriented according to $\mathcal{O}$. Then $c(G; \mathcal{O}) = |E(G)|$ if and only if $G_{\mathcal{O}}$ is the Hasse-diagram of some partial order.*

Proof. Suppose C is a cycle in G. Fix one of the two possible orientations of C and call an edge $e \in E(C)$ a forward (backward) edge if its orientation in $G_{\mathcal{O}}$ coincides (does not coincide) with the orientation of C. Clearly $c(G; \mathcal{O}) = |E(G)|$ if and only if each cycle in G contains at least two forward and two backward edges. This property, in turn, is well known to characterize Hasse-Diagrams (see [12, p. 108]). □

In 1978, Nešetřil and Rödl [11] proved that there exist graphs with arbitrarily large girth which cannot be oriented as Hasse-diagrams. In view of Proposition 6, this can be reformulated as follows.

Theorem 7 (Nešetřil and Rödl [11]). *For each natural number s, there exists a graph G with girth at least s and satisfying $c(G, \mathcal{O}) < |E(G)|$ for all acyclic orientations $\mathcal{O}$ of G.*

Our aim is to prove the following generalization of Theorem 7.

Theorem 8. *For each natural number s, there exists a nonexhaustive graph with girth at least s.*

Proof. The following statement is an easy corollary of the main theorem in [11].

Suppose $G = (V, E)$ is a graph with girth at least s and $\mathcal{O}$ an acyclic orientation of G. Then there exists a graph $G' = (V', E')$ with girth at least s such that for each acyclic orientation $\mathcal{O}'$ of G' there exists an orientation-preserving mapping $\phi : (G, \mathcal{O}) \to (G', \mathcal{O}')$ which maps G to an isomorphic subgraph of G'.

By checking the proof of Nešetřil and Rödl, we see that the graph G' is obtained as follows: First choose a certain $|V|$-uniform hypergraph $\mathscr{H}$ with vertex set V' and edge set $\mathscr{U}$ which does not contain cycles of length less than s. Then an isomorphic copy G_U of G is specified on each hyperedge $U \in \mathscr{U}$. E' is the union of all the edge sets $E(G_U)$, $U \in \mathscr{U}$.

Now choose as G with orientation $\mathcal{O}$ a directed path with $2s - 4$ edges and modify G' as follows: Choose a new vertex v_U for each $U \in \mathscr{U}$ and join it to the endpoints and to the middle point of the path G_U. The new graph G'' has the following properties:

(i) G'' has girth at least s and

(ii) G'' is not exhaustive.

Property (i) follows immediately from the construction. To see non-exhaustiveness, assume that Algy first tests all the edges of $G' \subset G''$. By the Nešetřil–Rödl result, there

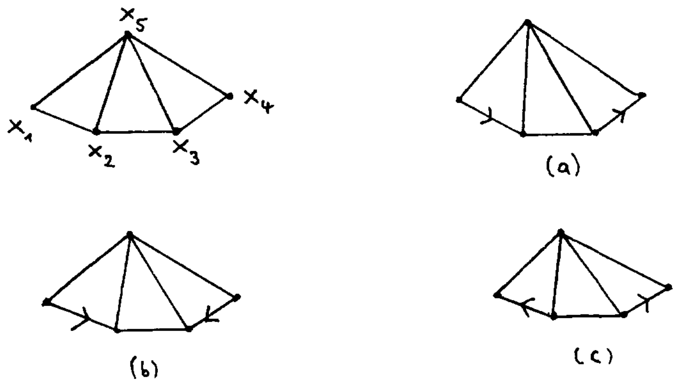

Fig. 1.

exists some $U \in \mathscr{U}$ such that G_U is a directed path with $2s - 4$ edges. By probing the edge between v_U and the middle-vertex of the path next, Algy will save one of the other two edges which are incident with v_U. □

Theorem 8 shows that exhaustiveness is *not* a local property. Nevertheless, the exhaustive *chordal* graphs can be characterized by excluding two forbidden subgraphs. It is well-known that K_4 is not exhaustive (4 numbers can be sorted by 5 comparisons!). Denote by G_0 the graph which is obtained by joining a vertex to all vertices of a path of length 3. Then G_0 is not exhaustive. To see this, number the vertices of G_0 as in Fig. 1 and ask first for the orientation of the edges $x_1 x_2$ and $x_3 x_4$. For reasons of symmetry we have to consider the three cases of Fig. 1 only.

In case (a), we ask x_3x_5. If the answer is $x_5 \to x_3$, the orientation of x_5x_4 is clear. In case $x_3 \to x_5$, we ask x_2x_5. If the answer is $x_2 \to x_5$, we save the edge x_1x_5, otherwise we save x_2x_3.

In case (b), we ask x_2x_3. W.l.o.g. assume that the answer is $x_2 \to x_3$. Then we ask x_2x_5 and save x_3x_5 or x_1x_5.

Finally, in case (c) we ask x_2x_3 and are essentially in the situation of case (b).

Proposition 9. *A chordal graph is exhaustive if and only if it does not contain K_4 or G_0 as an induced subgraph.*

Proof. Clearly, exhaustive graphs cannot contain K_4 or G_0 as an induced subgraph. On the other hand, a chordal graph G without a K_4 or G_0 has a very special structure: Call two triangles T_1 and T_2 equivalent if and only if they have an edge in common. Because of the absence of G_0 this is an equivalence relation and for each equivalence class $\mathscr{T}$ we can choose an edge $e_{\mathscr{T}}$ which is contained in all triangles of $\mathscr{T}$. The following strategy for player S is easily seen to force Algy to probe all the edges:

(i) Choose a fixed orientation $\vec{e_{\mathscr{T}}}$ for each edge $e_{\mathscr{T}}$. When $e_{\mathscr{T}}$ is tested during the game, it is oriented according to that orientation.

(ii) If an edge $f \neq e_{\mathscr{T}}$ of the triangle T, $T \in \mathscr{T}$, is probed for the first time, f is oriented such that $\vec{f}$ and $\vec{e_{\mathscr{T}}}$ do not form a directed path of length 2.

(iii) If for the second time an edge $f \neq e_{\mathscr{T}}$ of T, $T \in \mathscr{T}$, is probed, f is oriented such that $\vec{f}$ and $\vec{e_{\mathscr{T}}}$ form a directed path of length 2.

(iv) Edges which are not on a triangle are oriented arbitrarily.

Exhaustiveness can now be proved by induction, deleting a simplicial vertex of the chordal graph. $\square$

How many edges can an exhaustive graph have? The answer is given for $n \geqslant 6$ by the following theorem.

Theorem 10. (i) *For all $n \geqslant 6$, the number of edges in an exhaustive graph on n vertices is at most $\lfloor n^2/4 \rfloor$.*

(ii) *For all $n \geqslant 7$, the complete bipartite graph $K_{\lfloor n/2 \rfloor \lceil n/2 \rceil}$ is the only exhaustive graph with $\lfloor n^2/4 \rfloor$ edges.*

Proof. (i) Suppose first that (i) is true for some $n \geqslant 6$ and assume that there is an exhaustive graph $G = (V, E)$ on $n + 1$ vertices with $|E| = \lfloor (n+1)^2/4 \rfloor + 1$. If $n + 1$ is even, $n + 1 = 2k$, the average degree of G is $K + (1/k)$, hence there exists a vertex $x \in V$ of degree at most k. But the $G - x$ has at least $k^2 - k + 1 = \lfloor n^2/4 \rfloor + 1$ edges and is exhaustive too, a contradiction. Similarly, if $n + 1 = 2k + 1$, the average degree is $[2k(k+1) + 2]/(2k+1) < K + 1$ and the deletion of a vertex x of degree at most k yields an exhaustive graph with $2k$ vertices and at least $k^2 + 1$ edges, again a contradiction. It remains to prove the case $n = 6$. By checking the 15 nonisomorphic graphs with 6 vertices and 10 edges, e.g. by using the appendix of [6], we find that only two of the graphs do not contain K_4 or G_0, namely the graphs G_1 and G_2 depicted below (see Fig. 2). We leave it to the reader to specify algorithms proving that G_1 and G_2 are not exhaustive.

(ii) Again we apply induction on n. Suppose (ii) holds for some $n \geqslant 7$ and consider an exhaustive graph $G = (V, E)$ with $|V| = n + 1$ and $|E| = \lfloor (n+1)^2/4 \rfloor$. Suppose first that n is odd, $n + 1 = 2k$. By the validity of (i), the minimum degree of G is k. Since $|E(G)| = k^2$, G is k-regular. It follows from the induction hypothesis that $G - x$ is

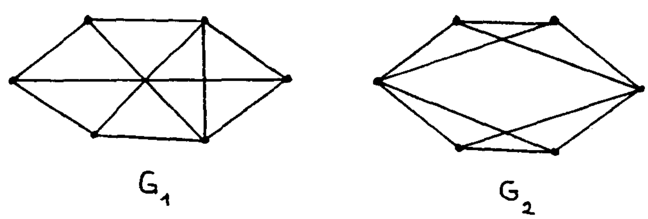

Fig. 2.

a complete bipartite graph for each $x \in V$. In particular, G does not contain a triangle and hence, in view of Turán's theorem, is isomorphic to $K_{k,k}$. The case of n even is similar. It is possible to complete the proof by checking the result for seven-vertex graphs, but since this is very tedious, we omit it and give a quick proof of (ii) for $n = 10$ instead. If there were an exhaustive graph G with 10 vertices and 25 edges, G not isomorphic to $K_{5,5}$, then G contains a triangle x, y, z. Since its minimum degree is 5, it is 5-regular and hence the exhaustive graph $G - \{x, y, z\}$ has 7 vertices and 13 edges which contradicts (i). $\square$

To conclude, we would like to mention two open problems:

1. What is the complexity status of deciding whether $c(G) \leqslant k$ for a graph G and some natural number k? We do not even know whether the problem is in $\mathcal{NP}$.
2. Is it true that $c(G) \leqslant (n^2/4) + o(n^2)$?

Note added in proof. The first version of our paper contained the question whether the line graphs of cubic triangle-free graphs are exhaustive. This has been disproved by Alon and Tuza.

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